Eur. Phys. J. B **29**, 339–343 (2002)
DOI: 10.1140/epjb/e2002-00313-8

THE EUROPEAN PHYSICAL JOURNAL B

Symmetry–breaking in local Lyapunov exponents

R. Ramaswamy[a]

School of Physical Sciences, Jawaharlal Nehru University, New Delhi 110 067, India

Received 25 October 2001 / Received in final form 8 December 2001
Published online 2 October 2002 –

Abstract. Integrable dynamical systems, namely those having as many independent conserved quantities as freedoms, have all Lyapunov exponents equal to zero. Locally, the instantaneous or finite time Lyapunov exponents are nonzero, but owing to a symmetry, their global averages vanish. When the system becomes nonintegrable, this symmetry is broken. A parallel to this phenomenon occurs in mappings which derive from quasiperiodic Schrödinger problems in 1–dimension. For values of the energy such that the eigenstate is extended, the Lyapunov exponent is zero, while if the eigenstate is localized, the Lyapunov exponent becomes negative. This occurs by a breaking of the quasiperiodic symmetry of local Lyapunov exponents, and corresponds to a breaking of a symmetry of the wavefunction in extended and critical states.

PACS. 05.45.-a Nonlinear dynamics and nonlinear dynamical systems – 05.45.Pq Numerical simulations of chaotic models – 71.30.+h Metal-insulator transitions and other electronic transitions

1 Introduction

The Lyapunov exponents (LEs) of a dynamical system characterize the manner in which elementary volumes in phase space evolve in time [1]. For an autonomous flow specified by the set of coupled differential equations

$$\dot{\mathcal{X}} = \mathcal{F}(\mathcal{X}), \qquad (\mathcal{X} \equiv X_1, \dots, X_k) \tag{1}$$

trajectories are obtained by evolving a given set of initial conditions $\mathcal{X}_0$. There are k Lyapunov exponents $\Lambda^m, m = 1, 2, \dots, k$ which are the logarithms of the eigenvalues of the matrix

$$\mathcal{O}(\mathcal{X}) = \lim_{N\to\infty} [\mathcal{M}(\mathcal{X}, N\delta t)^T \mathcal{M}(\mathcal{X}, N\delta t)]^{1/2N\delta t}, \tag{2}$$

where [2]

$$\mathcal{M}(\mathcal{X}, N\delta t) = \prod_{j=0}^{N-1} \mathcal{DF}(\mathcal{X}(j\delta t)), \tag{3}$$

$\mathcal{DF}$ is the Jacobian matrix, $\delta\mathcal{F}(\mathcal{X})/\delta\mathcal{X}$ and T denotes the transpose.

The Lyapunov exponents $\Lambda^1 \geq \Lambda^2 \geq \dots \geq \Lambda^k$ characterize the manner in which a k–dimensional parallelepiped evolves under the dynamics. Local (or finite time) Lyapunov exponents (LLE or FTLE), denoted $\lambda_N^m(\mathcal{X}_0), m = 1, 2, \dots, k$ are similarly defined for finite N in equation (2). Note that the λ_N^j's explicitly depend on initial conditions, $\mathcal{X}_0$, (although this will not always be indicated) while the asymptotic exponents, Λ^j's do not [3], and

$$\Lambda^m = \lim_{N\to\infty} \lambda_N^m(\mathcal{X}_0), \qquad m = 1, 2 \dots, k. \tag{4}$$

Finite–time LEs [4] offer a more detailed description of local dynamics and can be of considerable use in assessing the predictability of chaotic systems [5,6].

For discrete dynamical systems which are specified by iterative mappings,

$$\mathcal{X}(j+1) = \mathcal{F}(\mathcal{X}(j)), \tag{5}$$

the Lyapunov exponents can be computed by propagating k-dimensional orthonormal vectors, $\hat{\mathbf{e}}^m, m = 1, \dots, k$ in the tangent space, namely according to the dynamics

$$\mathbf{e}_j^m = \mathcal{DF}(\mathcal{X}(j)) \ \cdot \hat{\mathbf{e}}_{j-1}^m. \tag{6}$$

The vectors $\mathbf{e}_j^m$ when re-orthogonalized give the expansion (or contraction) rates along the different directions in phase space [7]. The FTLEs are, in this case,

$$\lambda_N^m = \frac{1}{N} \sum_{j=1}^{N} \ln \|\mathbf{e}_j^m\|, \qquad m = 1, 2, \dots k \tag{7}$$

and the spectrum of global LEs is obtained asymptotically as (*cf.* Eq. (4)) $\Lambda^m = \lim_{N\to\infty} \lambda_N^m, m = 1, 2, \dots k$.

The instantaneous Lyapunov exponent at the position $\mathcal{X}(j)$ along the trajectory, also termed the stretch exponent, is the quantity $\lambda_1^m(j)$. Regardless of whether a Lyapunov exponent is positive, negative or zero, LLEs can

[a] e-mail: `r.ramaswamy@mail.jnu.ac.in`

be positive or negative along a trajectory. The asymptotic value of the Lyapunov exponent thus depends on how the expanding (or positive LLE) and contracting (or negative LLE) regions are distributed along a trajectory. The distributions of FTLEs in both chaotic and nonchaotic systems have been extensively studied [5,6].

When is a Lyapunov exponent zero? What are the implications of a Lyapunov exponent being zero? These questions form the focus of the present paper, with specific application to nonintegrable Hamiltonian dynamical systems and maps that derive from a discrete Schrödinger equation.

For every quantity that is conserved in the dynamical system, there is at least one LE that is zero [8]. In Hamiltonian systems, conservation of the total phase volume implies that LEs come in pairs, so that each conserved quantity corresponds to two LEs being zero. Also, by Noether's theorem [9], a symmetry in the dynamics (for example invariance under time translation, say) also implies a zero Lyapunov exponent.

In an arbitrary dynamical system it appears unusual that all LEs can be zero. In general, this can occur in the following ways:

1. All the local LE's themselves are zero. This is a trivial case.
2. Stationarity holds: the sum of all negative instantaneous exponents is equal to the sum of all positive exponents.
3. Detailed balance holds: there is a pairwise cancellation of the instantaneous stretch exponents, say $\lambda_1^m(i) = -\lambda_1^m(j)$ for some i, j.

In the next section we discuss Cases 2 and 3 above in the context of several discrete quasiperiodic Schrödinger equations. These are the most interesting since the cancellation of the positive and negative contributions to the LE (See Eq. (2)) are *exact*, implying a symmetry in the system. Note however, that this symmetry is present in the tangent bundle rather than the phase space itself. In Section 3, analogous phenomena in integrable and nonintegrable Hamiltonian dynamical systems is studied, and the parallels in the two situations are elaborated and discussed in the final Section 4.

2 Quasiperiodic Schrödinger problems

The nature of the eigenvalues and eigenfunctions of the discrete Schrödinger equation,

$$\psi_{n+1} + \psi_{n-1} + V_n\psi_n = E\psi_n \tag{8}$$

where ψ_n is the wavefunction and V_n is the potential at lattice site n has been of interest for the past few decades [10,11]. If the potential V_n is periodic, then the eigenstates are Bloch states; the allowed energies form bands, and the wavefunctions are extended. When the potential is a random function of the lattice site index, then the spectrum is pure point, and the eigenfunctions are localized: they extend over a finite number of lattice sites, typically decaying exponentially from the maximum, $\psi_n \sim \psi_0 \exp(-n/\gamma)$ [12].

Through the transformation $x_n = \psi_{n-1}/\psi_n$, the above discrete Schrödinger equation becomes an equivalent iterative mapping [13–15],

$$x_{n+1} = \frac{-1}{x_n - E + V_n} \cdot \tag{9}$$

The potential term transforms into a "time–dependent" driving term. The linearity of the Schrödinger equation results in the invertibility of the mapping. It is easy to see that the Lyapunov exponent [16] of the above iterative map,

$$\Lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln x_{i+1}^2 \tag{10}$$

is essentially the inverse of the localization length, $\Lambda = -2/\gamma$, localized states of equation (8) corresponding to attractors of equation (9) with negative Lyapunov exponents. Extended and critical states have infinite localization length, and therefore $\Lambda = 0$.

A quasiperiodic potential is in some sense intermediate between periodic and random, and the nature of eigenstates of the system now depends on the values of parameters. In particular, if $V_n = 2\epsilon\cos(2\pi n\omega + \phi_0)$ with ω an irrational number, equation (8) is known as the Harper [17] or "almost Mathieu" [18] equation, and from a large body of work it is known that all states are extended below $\epsilon = 1$, all states are localized above $\epsilon = 1$ and at $\epsilon = 1$, all states are critical, namely they are localized as a *power–law* rather than exponentially.

In terms of the Lyapunov exponent of the corresponding iterative mapping, namely equation (9) with this V_n, the Lyapunov exponent is strictly zero below $\epsilon = 1$. Above this critical value, there are simple arguments to suggest that the dynamics is nontrivial, and that the attractors of this system are fractal [13]. At the same time, given the quasiperiodic nature of the dynamics, there are no periodic orbits and therefore, in general the system never recurs exactly. Thus, the result of a zero value for the Lyapunov exponent must follow from generalized symmetries.

2.1 Symmetry breaking in the localization transition

The metal–insulator transition in this system can be directly examined by following an eigenvalue as a function of the coupling parameter, ϵ. Shown in Figure 1 is the Lyapunov exponent for $E = 0$. The irrational number ω is chosen to be the golden mean ratio, $(\sqrt{5}-1)/2$ (though any Diophantine irrational number will show identical behaviour). The state goes from being extended for $\epsilon < 1$ to critical at $\epsilon = 1$, and is localized for $\epsilon > 1$, and thus the localization transition can be viewed as a bifurcation, and in previous work [14] we have shown that this transition is to a strange nonchaotic attractor [19,20].

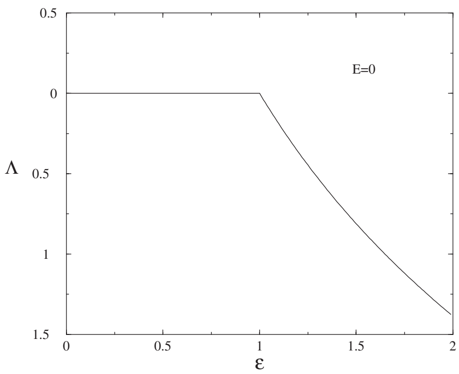

Fig. 1. Lyapunov exponent *versus* ϵ in the Harper map, equation (9) with $V_n = 2\epsilon \cos 2\pi\omega n + \phi_0$. At $\epsilon = 1$ there is a metal–insulator transition from a critical state to an exponentially localized state.

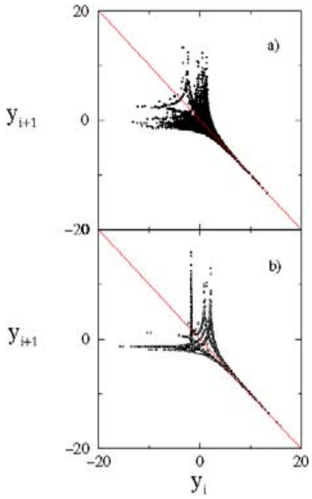

Fig. 2. Return map for the stretch exponent in the Harper system for (a) a critical state, $E = 0, \epsilon = 1$, which has the symmetry, (b) a localized state, $E = 0, \epsilon = 1.5$ which has no symmetry and results in a negative Lyapunov exponent.

The fact that the Lyapunov exponent, equation (10), is zero is nontrivial, and is a consequence of the analog of the KAM theory [10]. Here it can be seen as a consequence of a symmetry in the Harper map. Examination of the stretch exponents for the map, equation (9), which are $\lambda_1^1(x_i) = y_i = \ln x_{i+1}^2$ shows that there is an exact quasiperiodic symmetry. The return map, a plot of y_i *versus* y_{i+1} is shown in Figure 2. Up till $\epsilon = 1$ (see Fig. 1) the return map for the stretch exponent is *symmetric* with respect to the line $y = -x$, namely the discrete symmetry $(x, y) \to (-y, -x)$. Because of the quasiperiodicity there are no exactly symmetrically placed points except in the asymptotic limit. (It should be added that apart from $E = 0$, other eigenvalues are known only to finite (machine) precision; our numerical results are strongly suggestive of the exact symmetry but other than for $E = 0$, this is difficult to verify with high precision.)

The nature of this symmetry that gives rise to a zero value of the Lyapunov exponent for extended and critical states relates to a property of the wavefunctions. From Figure 2a, it is clear that along an orbit, every pair of stretch exponents, (y_i, y_{i+1}) is approximately matched by another pair (y_j, y_{j+1}) such that $y_i \approx -y_{j+1}$ and $y_{i+1} \approx -y_j$. From the definition of the stretch exponent it is easy to see that this translates into a condition on the ratio of the wavefunction at two sets of sites. Explicitly, if the wavefunction at three neighbouring sites are in the ratio

$$\psi_{i-1} : \psi_i : \psi_{i+1} = a : b : c$$

then there is another triple of sites such that the wavefunction is

$$\psi_{j-1} : \psi_j : \psi_{j+1} \approx c : b : a.$$

The relation between i and j depends on the nature of the irrational frequency ω.

Indeed, this symmetry is present in the FTLEs for any N as well, so that the constraint is actually stronger: the ratio of the wavefunctions at any $N + 1$ consecutive sites is reversed elsewhere in the infinite chain, namely

$$\psi_j : \psi_{j+1} \ldots : \psi_{j+N} \approx \psi_{i+N} : \ldots \psi_{i+1} : \psi_i$$

and because of the quasiperiodicity, this must happen infinitely often.

In the localized state, the above symmetry of the stretch exponents is broken; see Figure 2b. Wavefunctions now decay exponentially, although there is evidence of fractal fluctuations about the exponentially decaying envelope.

2.2 Critical localization

This constraint on the nature of the wavefunction amplitudes seems to be more general than in the case of the Harper system alone, and applies to a number of other quasiperiodic and aperiodic potentials which support critically (power–law) localized states. Apart from the Harper equation and its generalizations [21], some of these are the Kohmoto model [22] and potentials V_n that derive from abstract quasiperiodic or aperiodic sequences such as the Fibonacci, Thue–Morse or period doubling sequences [23].

For all these systems maps (*cf.* Eq. (9)) can be derived, and whenever E is an eigenvalue of the corresponding equation (8), then the Lyapunov exponent is again exactly zero. Examination of the return map for the instantaneous LLE in such cases shows, just as in the case of the Harper system, there is an exact quasiperiodic symmetry that causes the LE to vanish, both for the Fibonacci chain, which is quasiperiodic, (Fig. 3a) as well as for the Thue–Morse chain, which is aperiodic (Fig. 3b). The same holds for any finite time LE as well.

Thus, in the above set of examples, the Lyapunov exponent being zero implies a quasiperiodic symmetry, a consequence of which is the extended or weakly localized nature of the wavefunction. The transition to exponential localization is accompanied by the breaking of this symmetry, which gives nonzero values of the Lyapunov exponent.

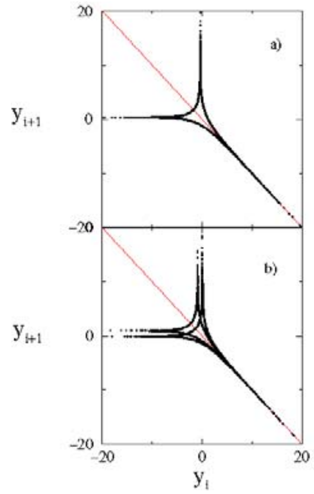

Fig. 3. Return map for the stretch exponent for critically localized states in (a) the Fibonacci chain and (b) the Thue–Morse chain.

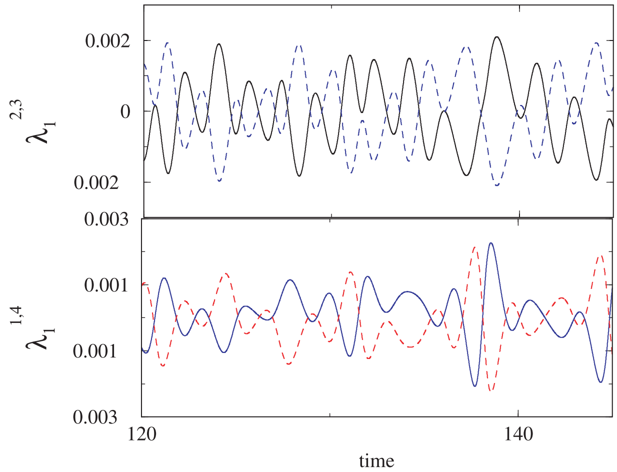

Fig. 4. Instantaneous Lyapunov exponents for a regular (torus) orbit of the Henon Heiles system. $\sum_{i=1}^{4} \Lambda^i = \sum_{i=1}^{4} \lambda_1^i(t) = 0$ and further, all the stretch exponents have the detailed balance quasiperiodic symmetry by which the global LEs are all zero.

3 Hamiltonian systems

We now examine a parallel situation in the case of conservative Hamiltonian flows. An autonomous Hamiltonian $\mathbf{H}(\mathbf{q},\mathbf{p})$ with k freedoms has the $2k$ equations of motion $(\mathcal{X} \equiv q_1,\ldots,q_k,p_1,\ldots,p_k)$

$$\dot{\mathcal{X}} = \mathbf{J}\frac{\partial \mathbf{H}}{\partial \mathcal{X}}, \tag{11}$$

in the standard symplectic notation [9]. The LEs must sum to zero both globally as well as locally, namely

$$\sum_{i=1}^{2k} \Lambda^i = 0 = \sum_{i=1}^{2k} \lambda_1^i(t) \quad \text{for any time } t \tag{12}$$

(the time dependence of the λ_1^i's is indicated, and is equivalent to evaluating them at each point in the phase space along an orbit). The symplectic structure gives the *pairing rule* for the LEs [8], $\Lambda^i + \Lambda^j = 0, \lambda_1^i(t) + \lambda_1^j(t) = 0$ for $j = 2k+1-i$. This together with the fact that time-translation invariance gives a zero LE, implies that the spectrum of LEs can be ordered as

$$\Lambda^1 \geq \Lambda^2 \geq \ldots \geq \Lambda^k = 0 = \Lambda^{k+1} \geq \Lambda^{k+2} \geq \ldots \geq \Lambda^{2k} = -\Lambda^1. \tag{13}$$

Equations (12, 13) hold for all Hamiltonian systems whether integrable or not. However, if the system is integrable, then all the Λ's are *individually* zero; this can only happen if there is a symmetry in the stretch exponents causing them to cancel exactly, namely $\lambda_1^i(\tau) = -\lambda_1^i(\tau')$ for some τ, τ'. This is equivalent to a detailed balance condition.

For integrable as well as nonintegrable systems wherein the KAM theory applies, so long as the motion is confined to KAM tori [24], this detailed balance must hold in order to give vanishing Lyapunov exponents. For nonintegrable dynamics, the symmetry is broken, leading to a nonzero average for some of the LEs.

A specific example is provided by the extensively studied Hénon-Heiles Hamiltonian [25]

$$\mathbf{H}(\mathbf{q},\mathbf{p}) = \frac{1}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2) + q_2^2 q_1 - \frac{1}{3}q_1^3. \tag{14}$$

In this system the nature of the motion depends on initial conditions since the system is quasi-integrable. So long as the motion is on tori, then by the KAM theorem, the dynamics is effectively integrable and all Λ's are zero. The dynamics is quasiperiodic and the stretch exponents are nonconstant, so that the global average being zero results from a cancellation of terms $\lambda_1^i(\tau) = -\lambda_1^i(\tau')$ for some τ, τ'. Further, these are symmetric and paired $(\lambda_1^1 = -\lambda_1^4, \lambda_1^2 = -\lambda_1^3$; see Fig. 4). Indeed, a plot of $\lambda_1^i(t)$ *versus* $\lambda_1^i(t+\delta t)$ namely a return map analogous to Figures 2 or 3 is symmetric with respect to the line $y = -x$, giving direct evidence of the detailed balance condition.

This detailed balance is lost when the motion is chaotic; there is no underlying torus, and although the pairing holds since all exponents should sum to zero (Fig. 5), the stretch exponents λ_1^1 and λ_1^4 are no longer symmetric, having nonzero averages. Note that $\Lambda^2 = \Lambda^3 = 0$ as in Figure 4, and for these exponents, the detailed balance condition continues to hold.

From the viewpoint of local Lyapunov exponents, therefore, the break-up of KAM tori and the transition from regular motion to chaos [16,24] is a symmetry–breaking transition as well.

4 Summary

The transition from regular to chaotic dynamics and the transition from extended to localized states, when examined in terms of the behaviour of Lyapunov exponents,

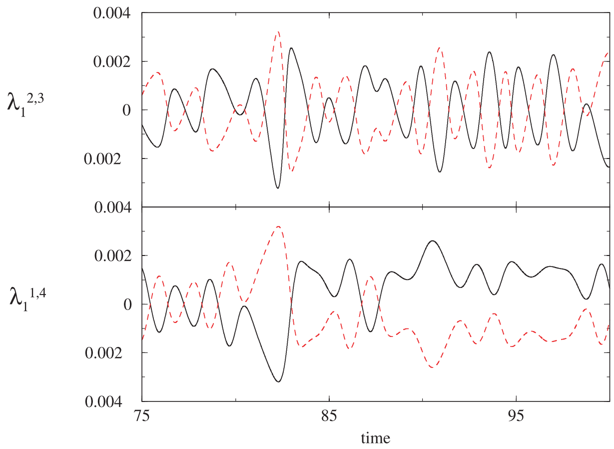

Fig. 5. As in Figure 4 for a chaotic orbit. The global Lyapunov exponents sum to zero, but the detailed balance symmetry of the local exponents λ_1^1 and λ_1^4 is broken, leading to a nonzero value for Λ^1 and Λ^4. Λ^2 and Λ^3 are zero since λ_1^2 and λ_1^3 have the symmetry.

show common features. In either case, there is a loss of a specific symmetry in the stretch (or local Lyapunov) exponents, resulting in a global Lyapunov exponent becoming nonzero.

In Hamiltonian dynamical systems, the conservation of phase–space volume requires that Lyapunov exponents come in pairs, each pair summing to zero. Integrability requires that each Lyapunov exponent is zero, and this happens by a symmetry in the local Lyapunov exponent which results in a zero average. When a given KAM torus breaks up, this transition from regular to chaotic dynamics, is signaled by a loss of this symmetry of instantaneous Lyapunov exponents, leading to a nonzero value for the averages.

The same loss of symmetry in the instantaneous Lyapunov exponent is seen in the dynamics of a map which is equivalent to a discrete quasiperiodic Schrödinger equation when one examines the transition from extended to localized states. The implications of the symmetry is that in extended and critically localized states, the amplitude ratio of wave–functions at (an arbitrary number of) N neighboring sites is approximately mirrored in reverse order at an infinite number of sets of other N neighboring sites. This poses a *severe* constraint on the wavefunction amplitudes, consistent with the lack of (or weak) decay of amplitude in extended (or critical) states.

The present paper is an initial step in examining the similarities of these two different transition scenarios. Some of the related issues which we hope to explore in future work are an examination of local Lyapunov exponents in the transition from KAM tori to cantori [26], and the generality of symmetry–breaking in the localization transition in other models displaying critical localization [23].

This work is supported by a grant from the Department of Science and Technology. I would like to thank the CSIR for travel support, and N. Bhatnagar, S. Datta, S. Negi, A. Prasad, and I. Satija for collaboration on various aspects of this work.

References

1. J.P. Eckmann, D. Ruelle, Rev. Mod. Phys. **57**, 617 (1985)
2. C. Ziehmann, L.A. Smith, J. Kurths, Physica D **126**, 49 (1999)
3. V.I. Oseledec, Trans. Moscow Math. Soc. **19**, 197 (1969)
4. H.D.I. Abarbanel, R. Brown, M.B. Kennel, J. Nonlinear Sci. **1**, 175 (1991)
5. P. Grassberger, R. Badii, A. Politi, J. Stat. Phys. **51**, 135 (1988)
6. A. Prasad, R. Ramaswamy, Phys. Rev. E **60**, 2761 (1999); in *Nonlinear Dynamics: Integrability and Chaos*, edited by M. Daniel, K.M. Tamizhmani, R. Sahadevan, Narosa Publishing House, New Delhi, 2000) pp. 227
7. G. Bennetin, L. Galgani, J.-M. Strelcyn, Phys. Rev. A **14**, 2338 (1976)
8. H.-D. Meyer, J. Chem. Phys. **84**, 3147 (1986)
9. See, *e.g.*, H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, 1980)
10. G. André, S. Aubry, Ann. Isr. Phys. Soc. **3**, 133 (1980)
11. D.R. Hofstadter, Phys. Rev. B **14**, 2239 (1976); M. Kohmoto, L.P. Kadanoff, C. Tang, Phys. Rev. Lett. **50**, 1870 (1983); S. Ostlund, R. Pandit, D. Rand, H.J. Schellnhuber, E.D. Siggia, Phys. Rev. Lett. **50**, 1873 (1983); J.B. Sokoloff, Phys. Rep. **126**, 189 (1985); T. Geisel, R. Ketzmerick, G. Petschel, Phys. Rev. Lett. **66**, 1651 (1991)
12. In addition there can be critically localized states which decay as a power law rather than as an exponential: these have infinite localization length
13. J. Ketoja, I. Satija, Physica D **109**, 70 (1997)
14. A. Prasad, R. Ramaswamy, I. Satija, N. Shah, Phys. Rev. Lett. **83**, 4530 (1999)
15. S.S. Negi, R. Ramaswamy, Phys. Rev. E **64**, 045204R (2001)
16. E. Ott, *Chaos in dynamical systems* (Cambridge University Press, Cambridge, UK, 1994)
17. P.G. Harper, Proc. Phys. Soc. London A **68**, 874 (1955)
18. Y. Last, in *XI Intern. Congress of Mathematical Physics (Paris, 1994)*, edited by D. Iagolnitzer (Intern. Press, Cambridge, MA, 1995), pp. 366–372; S. Jitomirskaya, *ibid.*, pp. 373-382; S.Y. Jitomirskaya, Y. Last, Phys. Rev. Lett. **76**, 1765 (1996)
19. C. Grebogi, E. Ott, S. Pelikan, J.A. Yorke, Physica D **13**, 261 (1984)
20. A. Prasad, S.S. Negi, R. Ramaswamy, Int. J. Bifurc. Chaos, **11**, 291 (2001)
21. H. Hiramoto, M. Kohmoto, Int. J. Mod. Phys. B **6**, 281 (1992)
22. M. Kohmoto, Phys. Rev. Lett. **51**, 1198 (1983)
23. J.M. Luck, Phys. Rev. B **39**, 5834 (1989)
24. M. Tabor, *Chaos and integrability in nonlinear dynamics* (John Wiley and Sons, New York, 1989)
25. M. Hénon, C. Heiles, Astron. J. **69**, 73 (1964)
26. R.S. Mackay, J.D. Meiss, I.C. Percival, Phys. Rev. Lett. **52**, 697 (1984); Physica D **13**, 55 (1984)